

The $\mathcal{O}\left(\frac{\alpha_{em}}{\alpha_s}\right)$ correction to $BR[B \rightarrow X_s \gamma]$

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Abstract

We evaluate the $\mathcal{O}\left(\frac{\alpha_{em}}{\alpha_s}\right)$ correction to the rate of $B \rightarrow X_s \gamma$ decay, i.e. we resum all the $\mathcal{O}[(\alpha_{em} \ln M_W^2/m_b^2) \times (\alpha_s \ln M_W^2/m_b^2)^n]$ corrections for $n = 0, 1, 2, \dots$. Our calculation differs from the previously available one by that it takes into account the complete relevant set of operators. The correction is found to be negligible, i.e. it is below 1%, in accordance with the former results.

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At present, the next-to-leading logarithmic QCD analysis of $B \rightarrow X_s \gamma$ decay allows predicting its branching ratio with around 10% accuracy [1]. The current accuracy on the experimental side is around 15% [2], which is expected to be improved soon. Therefore, it is important to examine the size of the dominant electroweak corrections to this decay mode.

Prior to an explicit calculation, two sets of electroweak corrections are expected to be dominant. The first set consists of corrections that are enhanced by the ratio m_t^2/M_W^2 . Such contributions have been evaluated in ref. [3] and found to have less than 1% effect on the branching ratio, owing to an accidental cancellation. The second set consists of corrections that are enhanced by the large logarithm $\ln M_W^2/m_b^2$. The authors of ref. [4] have calculated them and found a contribution of only around 1% to the branching ratio.

However, when the $\mathcal{O}(\alpha_{em} \ln M_W^2/m_b^2)$ correction is small, it might happen that some of the $\mathcal{O}[(\alpha_{em} \ln M_W^2/m_b^2) \times (\alpha_s \ln M_W^2/m_b^2)^n]$ corrections are much bigger, while only the $n = 0$ case was included in ref. [4]. Since $\alpha_s \ln M_W^2/m_b^2$ is close to unity, a naive estimate for the sum of all the $\mathcal{O}[(\alpha_{em} \ln M_W^2/m_b^2) \times (\alpha_s \ln M_W^2/m_b^2)^n]$ corrections is $\frac{\alpha_{em}}{\alpha_s(M_W)} \times (\text{number of order unity}) = 6.5\% \times (\text{number of order unity})$. The above estimate is close to the overall $\mathcal{O}(10\%)$ uncertainty of the prediction for the branching ratio. Thus, there is a good reason for performing a complete $\mathcal{O}(\frac{\alpha_{em}}{\alpha_s})$ calculation.

Resummation of the large logarithms $\ln M_W^2/m_b^2$ from all orders of the perturbation series is most conveniently performed in the framework of an effective theory that is obtained from the Standard Model by decoupling the top quark and the heavy electroweak bosons.

The part of the effective theory lagrangian that is relevant to $b \rightarrow s \gamma$ reads

$$\mathcal{L}_{eff} = \mathcal{L}_{QCD \times QED}(u, d, s, c, b) + \frac{4G_F}{\sqrt{2}} V_{ts}^* V_{tb} \left[\sum_{i=1}^8 C_i(\mu) P_i(\mu) + \sum_{i=3}^6 C_i^Q(\mu) P_i^Q(\mu) \right], \quad (1)$$

where V_{ij} are elements of the CKM matrix, while $C_i(\mu)$ and $C_i^Q(\mu)$ are the Wilson coefficients at the following operators:

$$\begin{aligned} P_1 &= (\bar{s}_L \gamma_\mu T^a c_L)(\bar{c}_L \gamma^\mu T^a b_L), \\ P_2 &= (\bar{s}_L \gamma_\mu c_L)(\bar{c}_L \gamma^\mu b_L), \\ P_3 &= (\bar{s}_L \gamma_\mu b_L) \sum_q (\bar{q} \gamma^\mu q), \\ P_4 &= (\bar{s}_L \gamma_\mu T^a b_L) \sum_q (\bar{q} \gamma^\mu T^a q), \\ P_5 &= (\bar{s}_L \gamma_{\mu_1} \gamma_{\mu_2} \gamma_{\mu_3} b_L) \sum_q (\bar{q} \gamma^{\mu_1} \gamma^{\mu_2} \gamma^{\mu_3} q), \\ P_6 &= (\bar{s}_L \gamma_{\mu_1} \gamma_{\mu_2} \gamma_{\mu_3} T^a b_L) \sum_q (\bar{q} \gamma^{\mu_1} \gamma^{\mu_2} \gamma^{\mu_3} T^a q), \\ P_3^Q &= (\bar{s}_L \gamma_\mu b_L) \sum_q Q_q (\bar{q} \gamma^\mu q), \\ P_4^Q &= (\bar{s}_L \gamma_\mu T^a b_L) \sum_q Q_q (\bar{q} \gamma^\mu T^a q), \end{aligned}$$

$$\begin{aligned}
P_5^Q &= (\bar{s}_L \gamma_{\mu_1} \gamma_{\mu_2} \gamma_{\mu_3} b_L) \sum_q Q_q (\bar{q} \gamma^{\mu_1} \gamma^{\mu_2} \gamma^{\mu_3} q), \\
P_6^Q &= (\bar{s}_L \gamma_{\mu_1} \gamma_{\mu_2} \gamma_{\mu_3} T^a b_L) \sum_q Q_q (\bar{q} \gamma^{\mu_1} \gamma^{\mu_2} \gamma^{\mu_3} T^a q), \\
P_7 &= \frac{e}{16\pi^2} m_b (\bar{s}_L \sigma^{\mu\nu} b_R) F_{\mu\nu}, \\
P_8 &= \frac{g}{16\pi^2} m_b (\bar{s}_L \sigma^{\mu\nu} T^a b_R) G_{\mu\nu}^a.
\end{aligned} \tag{2}$$

The small CKM matrix element V_{ub} as well as the s -quark mass are neglected here.

The above set of operators closes under the QCD and QED renormalizations.² It is the QED renormalization that forces us to introduce the operators P_k^Q , in which sums over flavours are weighted by their electric charges. These operators were absent in the QCD analysis of ref. [1], because $\mathcal{O}(\frac{\alpha_{em}}{\alpha_s})$ corrections were neglected there.

In order to evaluate the $\mathcal{O}(\frac{\alpha_{em}}{\alpha_s})$ terms, one has to calculate the anomalous dimension matrix for all the above operators up to order $\mathcal{O}(\alpha_{em})$, and then solve the Renormalization Group Equations (RGEs). Such a calculation has already been performed in ref. [5]. However, the operator basis was truncated there to $\{P_1, P_2, P_7, P_8\}$ only.

The main purpose of the present paper is a complete evaluation of the $\mathcal{O}(\frac{\alpha_{em}}{\alpha_s})$ corrections, i.e. including all the relevant operators from eq. (2). At the same time, we shall check the results of ref. [5] and verify whether truncating the operator basis was a good approximation there.

Evaluating the anomalous dimension matrix at $\mathcal{O}(\alpha_s)$ and $\mathcal{O}(\alpha_{em})$ proceeds in full analogy to the well-known calculations of the leading-logarithmic QCD effects in $B \rightarrow X_s \gamma$ (see [6] and references therein). One needs to find the one-loop mixing among the four-quark operators, the one-loop mixing in the $\{P_7, P_8\}$ sector, as well as the two-loop mixing of the four-quark operators into P_7 and P_8 . Diagrams with virtual photons need to be included as well.

As in the case of the former QCD analyses [6, 1], it is convenient to introduce the so-called “effective coefficients” before the RGEs are solved. They are given by the following linear combinations of the original Wilson coefficients:

$$C_i^{eff}(\mu) = \begin{cases} C_7(\mu) + \sum_{i=3}^6 y_i \left[C_i(\mu) - \frac{1}{3} C_i^Q(\mu) \right], & \text{for } i = 7, \\ C_8(\mu) + \sum_{i=3}^6 z_i \left[C_i(\mu) - \frac{1}{3} C_i^Q(\mu) \right], & \text{for } i = 8, \\ C_i(\mu), & \text{otherwise.} \end{cases} \tag{3}$$

The numbers y_i and z_i are defined so that the leading and the $\mathcal{O}(\frac{\alpha_{em}}{\alpha_s})$ contributions to the $b \rightarrow s \gamma$ and $b \rightarrow s$ gluon matrix elements of the effective hamiltonian are proportional to

² It closes off-shell, up to non-physical operators that either vanish in four dimensions or vanish by the $QCD \times QED$ equations of motion. The existence of leptons is ignored here, because their effect $b \rightarrow s \gamma$ is of higher order in QED.

the corresponding terms in C_7^{eff} and C_8^{eff} , respectively. In dimensional regularization with fully anticommuting γ_5 , we have $y = (0, 0, -\frac{1}{3}, -\frac{4}{9}, -\frac{20}{3}, -\frac{80}{9})$ and $z = (0, 0, 1, -\frac{1}{6}, 20, -\frac{10}{3})$.

The effective coefficients evolve according to their RGE:³

$$\mu \frac{d}{d\mu} C_i^{eff}(\mu) = C_j^{eff}(\mu) \gamma_{ji}^{eff}(\mu) \quad (4)$$

driven by the anomalous dimension matrix $\hat{\gamma}^{eff}(\mu)$

$$\hat{\gamma}^{eff}(\mu) = \frac{\alpha_s(\mu)}{4\pi} \hat{\gamma}_s^{(0)eff} + \frac{\alpha_{em}}{4\pi} \hat{\gamma}_{em}^{(0)eff} + \dots \quad (5)$$

The matrices $\hat{\gamma}_s^{(0)eff}$ and $\hat{\gamma}_{em}^{(0)eff}$ are regularization- and renormalization-scheme independent, contrary to the matrices governing the evolution of the original coefficients $C_i(\mu)$.

Our results for $\hat{\gamma}_s^{(0)eff}$ and $\hat{\gamma}_{em}^{(0)eff}$ are the following:

	P_1	P_2	P_3	P_4	P_5	P_6	P_3^Q	P_4^Q	P_5^Q	P_6^Q	P_7	P_8	
$\hat{\gamma}_s^{(0)eff} =$	-4	$\frac{8}{3}$	0	$-\frac{2}{9}$	0	0	0	0	0	0	$-\frac{208}{243}$	$\frac{173}{162}$	P_1
	12	0	0	$\frac{4}{3}$	0	0	0	0	0	0	$\frac{416}{81}$	$\frac{70}{27}$	P_2
	0	0	0	$-\frac{52}{3}$	0	2	0	0	0	0	$-\frac{176}{81}$	$\frac{14}{27}$	P_3
	0	0	$-\frac{40}{9}$	$-\frac{100}{9}$	$\frac{4}{9}$	$\frac{5}{6}$	0	0	0	0	$-\frac{152}{243}$	$-\frac{587}{162}$	P_4
	0	0	0	$-\frac{256}{3}$	0	20	0	0	0	0	$-\frac{6272}{81}$	$\frac{6596}{27}$	P_5
	0	0	$-\frac{256}{9}$	$\frac{56}{9}$	$\frac{40}{9}$	$-\frac{2}{3}$	0	0	0	0	$\frac{4624}{243}$	$\frac{4772}{81}$	P_6
	0	0	0	$-\frac{8}{9}$	0	0	0	-20	0	2	$\frac{176}{243}$	$-\frac{14}{81}$	P_3^Q
	0	0	0	$\frac{16}{27}$	0	0	$-\frac{40}{9}$	$-\frac{52}{3}$	$\frac{4}{9}$	$\frac{5}{6}$	$-\frac{136}{729}$	$-\frac{295}{486}$	P_4^Q
	0	0	0	$-\frac{128}{9}$	0	0	0	-128	0	20	$\frac{6272}{243}$	$-\frac{764}{81}$	P_5^Q
	0	0	0	$\frac{184}{27}$	0	0	$-\frac{256}{9}$	$-\frac{160}{3}$	$\frac{40}{9}$	$-\frac{2}{3}$	$\frac{39152}{729}$	$-\frac{1892}{243}$	P_6^Q
	0	0	0	0	0	0	0	0	0	0	$\frac{32}{3}$	0	P_7
	0	0	0	0	0	0	0	0	0	0	$-\frac{32}{9}$	$\frac{28}{3}$	P_8
$\hat{\gamma}_{em}^{(0)eff} =$	$-\frac{8}{3}$	0	0	0	0	0	$\frac{32}{27}$	0	0	0	$-\frac{832}{729}$	$\frac{22}{243}$	P_1
	0	$-\frac{8}{3}$	0	0	0	0	$\frac{8}{9}$	0	0	0	$-\frac{208}{243}$	$-\frac{116}{81}$	P_2
	0	0	0	0	0	0	$\frac{76}{9}$	0	$-\frac{2}{3}$	0	$-\frac{20}{243}$	$\frac{20}{81}$	P_3
	0	0	0	0	0	0	$-\frac{32}{27}$	$\frac{20}{3}$	0	$-\frac{2}{3}$	$-\frac{176}{729}$	$\frac{14}{243}$	P_4
	0	0	0	0	0	0	$\frac{496}{9}$	0	$-\frac{20}{3}$	0	$-\frac{22712}{243}$	$\frac{1328}{81}$	P_5
	0	0	0	0	0	0	$-\frac{512}{27}$	$\frac{128}{3}$	0	$-\frac{20}{3}$	$-\frac{6272}{729}$	$-\frac{1180}{243}$	P_6
	0	0	0	0	0	0	0	0	0	0	$\frac{16}{9}$	$-\frac{8}{3}$	P_7
	0	0	0	0	0	0	0	0	0	0	0	$\frac{8}{9}$	P_8

³ In eqs. (4)–(9), we do not distinguish between C_i and C_i^Q , i.e. we write these equations as if the operators were numbered from 1 to 12.

Details of their evaluation can be found in ref. [7].

The rows corresponding to P_3^Q, \dots, P_6^Q in the matrix $\hat{\gamma}_{em}^{(0)eff}$ have not been given explicitly above. They would be relevant only at higher orders in α_{em} , because the coefficients of these operators start at $\mathcal{O}(\frac{\alpha_{em}}{\alpha_s})$.

In the above matrices, the QED mixing of P_3, \dots, P_6 into P_7 and P_8 as well as the QCD mixing of P_3^Q, \dots, P_6^Q into P_7 and P_8 are given for the first time. As far as the remaining entries are concerned, our results agree with the old ones of refs. [8, 6] and [5]. However, in order to perform a comparison, one needs to make a linear transformation of our matrices to the “old” basis of the four-quark operators used in those articles.

The solution of the RGE (4) with initial conditions at $\mu = M_W$ has the following form:

$$C_i^{eff}(\mu) = C_i^{(0)eff}(\mu) + \frac{\alpha_{em}}{\alpha_s(\mu)} C_i^{em(0)eff}(\mu) + \dots, \quad (6)$$

where

$$C_i^{(0)eff}(\mu) = \sum_{k,j} V_{ik} \eta^{a_k} V_{kj}^{-1} C_j^{(0)eff}(M_W), \quad (7)$$

$$C_i^{em(0)eff}(\mu) = \frac{1}{2\beta_0} \sum_{k,l,j} \frac{\eta^{a_l} - \eta^{a_k-1}}{1 - a_k + a_l} V_{ik} \left(\hat{V}^{-1} (\hat{\gamma}_{em}^{(0)eff})^T \hat{V} \right)_{kl} V_{lj}^{-1} C_j^{(0)eff}(M_W), \quad (8)$$

$\beta_0 = \frac{23}{3}$ and $\eta = \frac{\alpha_s(M_W)}{\alpha_s(\mu)}$. The matrix \hat{V} and the numbers a_k are obtained via diagonalization of $(\hat{\gamma}_s^{(0)eff})^T$

$$\left(\hat{V}^{-1} (\hat{\gamma}_s^{(0)eff})^T \hat{V} \right)_{kl} = 2\beta_0 a_k \delta_{kl}. \quad (9)$$

The above solution to the RGE is identical to the one found in refs. [8] and [5].⁴

Note that the $\mathcal{O}(\frac{\alpha_{em}}{\alpha_s})$ terms in the Wilson coefficients vanish at the initial scale $\mu = M_W$. It must be so, because no such terms can arise from perturbative matching of the SM and the effective theory amplitudes. The relevant matching conditions are thus the same as the leading-order ones in ref. [1]:

$$C_i^{(0)eff}(M_W) = C_i^{(0)}(M_W) = \begin{cases} 0, & \text{for } i = 1, 3, 4, 5, 6, \\ 1, & \text{for } i = 2, \\ \frac{3x^3 - 2x^2}{4(x-1)^4} \ln x + \frac{-8x^3 - 5x^2 + 7x}{24(x-1)^3}, & \text{for } i = 7, \\ \frac{-3x^2}{4(x-1)^4} \ln x + \frac{-x^3 + 5x^2 + 2x}{8(x-1)^3}, & \text{for } i = 8, \end{cases} \quad (10)$$

$$C_i^{Q(0)eff}(M_W) = 0,$$

where $x = m_t^2/M_W^2$.

⁴ Except for the misprint $\eta_i \leftrightarrow \eta_j$ in eq. (A.10) of ref. [5]

After substituting the explicit anomalous dimension matrices and the initial conditions to the solution of the RGE, we find

$$C_7^{(0)eff}(\mu_b) = \eta^{\frac{16}{23}} C_7^{(0)}(M_W) + \frac{8}{3} \left(\eta^{\frac{14}{23}} - \eta^{\frac{16}{23}} \right) C_8^{(0)}(M_W) + \sum_{i=1}^8 h_i \eta^{a_i}, \quad (11)$$

$$\begin{aligned} C_7^{em(0)eff}(\mu_b) &= \left(\frac{88}{575} \eta^{\frac{16}{23}} - \frac{40}{69} \eta^{-\frac{7}{23}} + \frac{32}{75} \eta^{-\frac{9}{23}} \right) C_7^{(0)}(M_W) \\ &+ \left(-\frac{704}{1725} \eta^{\frac{16}{23}} + \frac{640}{1449} \eta^{\frac{14}{23}} + \frac{32}{1449} \eta^{-\frac{7}{23}} - \frac{32}{575} \eta^{-\frac{9}{23}} \right) C_8^{(0)}(M_W) \\ &- \frac{526074716}{4417066408125} \eta^{-\frac{47}{23}} + \frac{65590}{1686113} \eta^{-\frac{20}{23}} + \sum_{i=1}^8 \left(h'_i \eta^{a_i} + h''_i \eta^{a_i-1} \right), \quad (12) \end{aligned}$$

with the values of a_i , h_i , h'_i and h''_i given in table 1.

i	a_i	h_i	h'_i	h''_i
1	$\frac{14}{23}$	$\frac{626126}{272277}$	$\frac{50090080}{131509791}$	$\frac{10974039505456}{21104973066375}$
2	$\frac{16}{23}$	$-\frac{56281}{51730}$	$-\frac{107668}{646625}$	$-\frac{13056852574}{29922509799}$
3	$\frac{6}{23}$	$-\frac{3}{7}$	$-\frac{3254504085930274}{23167509579260865}$	$-\frac{718812}{6954395}$
4	$-\frac{12}{23}$	$-\frac{1}{14}$	$\frac{34705151}{143124975}$	$-\frac{154428730}{12196819523}$
5	0.4086	-0.6494	-0.2502	-0.1374
6	-0.4230	-0.0380	0.1063	-0.0078
7	-0.8994	-0.0186	-0.0525	-0.0023
8	0.1456	-0.0057	0.0213	-0.0001

Table 1. The numbers a_i , h_i , h'_i and h''_i entering eqs. (11) and (12).

Setting μ_b to 5 GeV and taking the remaining parameters from ref. [9], one finds $\eta \simeq 0.56$, $m_t^2/M_W^2 \simeq 4.7$ and $\alpha_{em}/\alpha_s(\mu_b) \simeq 0.036$, which implies

$$C_7^{(0)eff}(\mu_b) = -0.313, \quad (13)$$

$$C_7^{em(0)eff}(\mu_b) = 0.033, \quad (14)$$

and, in consequence,

$$\frac{\Delta BR[B \rightarrow X_s \gamma]}{BR[B \rightarrow X_s \gamma]} = 2 \frac{\alpha_{em}}{\alpha_s(\mu_b)} \frac{C_7^{em(0)eff}(\mu_b)}{C_7^{(0)eff}(\mu_b)} \simeq -0.8\%. \quad (15)$$

The above $\mathcal{O}(\frac{\alpha_{em}}{\alpha_s})$ correction to $BR[B \rightarrow X_s \gamma]$ appears to be much smaller than the simple estimate presented at the beginning of this article. The structure of eqs. (11) and (12) leads to a naive expectation that $C_7^{(0)eff}(\mu_b)$ and $C_7^{em(0)eff}(\mu_b)$ are similar in magnitude. However, after the numerical evaluation, one of them turns out to be almost ten times smaller.

Although our analytical result for $C_7^{em(0)eff}(\mu_b)$ is different from eq. (11) in ref. [5], the two formulae are numerically very close, for realistic values of η . Thus, truncating the operator basis is a correct approximation in the present case.

To conclude: We have performed a complete calculation of the $\mathcal{O}(\frac{\alpha_{em}}{\alpha_s})$ correction to the branching ratio of $B \rightarrow X_s \gamma$. The correction is found to be approximately equal to -0.8% , i.e. negligibly small. However, without an explicit calculation, a correction several times larger could not have been excluded.

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